A hierarchical percolation model: some exact results on the Sierpinski gasket

1989 J. Phys. A: Math. Gen. 22 4477

(http://iopscience.iop.org/0305-4470/22/20/023)

J. Phys. A: Math. Gen. 22 (1989) 4477-4486. Printed in the UK

A hierarchical percolation model: some exact results on the Sierpinski gasket

Harry Fried

Department of Physics FM-15, University of Washington, Seattle, WA 98195, USA

Received 20 December 1988

Abstract. A hierarchical percolation model is formulated on the Sierpinski gasket. With the use of position space renormalisation group methods, exact results are obtained that place this model in a distinct universality class from that of the geometric percolation model. For the first of two cases considered, $p_c = 1$, $\nu = 1/D$ and for the second, $p_c = \frac{1}{2}$, $\nu = 1/(D-1)$, where D is the fractal dimension of the Sierpinski gasket. The hierarchical model is also formulated on the triangular lattice and some preliminary results are obtained.

1. Introduction

This work is motivated by the desire to understand the relevant properties that determine the universality class of a given percolation model. In this paper we consider a hierarchical percolation model defined on the Sierpinski gasket. Two distinct realisations of this model are studied using position space renormalisation group methods [1]. Exact results are found for the critical concentrations and correlation length exponents. These results show that the two models are contained in different universality classes, each of which is distinct from the universality class of the geometric percolation model [2]. The latter is known to be contained in the universality class of the ferromagnetic $q = 1$ Potts model [3, 4]. In addition, the hierarchical model is formulated on the triangular lattice and some of its properties are discussed.

2. Percolation models

A percolation model is defined as follows. At each site of a lattice place a site occupation variable; $n_i = 1$ if the site is occupied, $n_i = 0$ if the site is vacant. With each specific configuration of occupied sites, associate a statistical weight $W(\{n_i\}, p)$. The parameter p is the concentration of occupied sites. For the random occupation models that we consider here, this weight factor has the explicit form

$$W(\{n_i\}, p) = p^{n_s}(1-p)^{N-n_s}.$$

N is the total number of lattice sites and $n_s = \Sigma_i n_i$ is the total number of occupied

0305-4470/89/204477+10$02.50

sites for a given configuration. More generally, correlations between the sites can also be included. In addition define a mapping $\phi(R) \rightarrow C$ where R is a specific configuration of occupied sites and C is a partition of these sites into percolation clusters $\{C_i\}$. The mapping ϕ is explicitly defined by specifying a set of connection rules that determine the conditions under which two occupied sites are considered to be in the same cluster.

It is the average properties of these clusters (the averages are taken over all weighted configurations) which are of interest in the study of a percolation model. A percolation transition occurs at a concentration p_c at which, on average, there first occurs a cluster of infinite size. This transition is continuous if the average size of the largest cluster diverges continuously as p_c is approached from below. It is important to stress that the structure of the mapping ϕ defines the random occupation model. Therefore the universality class of a given percolation model is associated with the relevant properties of this mapping.

As an example let us consider the geometric percolation model. Here the connection rules are exceedingly simple. The basic rule is that two occupied nearest-neighbour sites are defined to be in the same cluster. Repeated application of this rule allows the construction of all clusters for a given configuration. This rule is *local* in that a site is added to a growing cluster if at least one of its nearest neighbours is in that cluster. No further constraints are necessary. By an exact mapping [3, 4] this model is known to be in the universality class of the ferromagnetic $q = 1$ Potts universality class.

The directed [5, 6] and bootstrap [7, 8] models are related to the geometric model. The directed model possesses an anisotropic version of the isotropic geometric rule. This anisotropy is relevant in that the critical properties of the directed model are distinct from those of the geometric model [6]. In the bootstrap model, an occupied site is considered *isolated* (i.e. contained in a cluster of size one) if it has fewer than m occupied *non-isolated* nearest neighbours. Each value of m defines a distinct bootstrap model. The local rule of the geometric model acts only between non-isolated occupied sites. The geometric model is therefore equivalent to the $m = 1$ bootstrap model. On the Bethe lattice the value of m is known to be relevant [7]. On Bravais lattices the value of m is irrelevant if the transition is continuous [8]; however, for specific choices of m the transition may be driven first order. An example of the latter case is the $m = z$ bootstrap models. (z is the coordination number of the lattice.) For $p < 1$ all occupied sites are isolated and so all clusters are of unit size. For $p = 1$ there is one infinite percolating cluster.

Recently a class of antiferromagnetic percolation models has been proposed and studied [9-12]. For these models, two occupied nearest-neighbour sites need not be in the same cluster; additional constraints are necessary for them to coalesce. These additional constraints lead to the possibility of non-local connection rules [9, 11] and to models exhibiting *two* distinct transitions [12]. A study of the critical properties of these models, however, has shown that all observed transitions are in the same universality class as the geometric model. Fried and Schick [12] have argued that this result can be traced to the generation of an effective geometric rule that becomes manifest under a prefacing transformation. This idea was implicit in the upper bound calculation of Adler *et al* [9].

The hierarchical model, to be defined below, has been devised in an attempt to suppress the generation of geometric percolation structures. On the Sierpinski gasket this suppression is complete. As we will see, on the triangular lattice the situation is more complex. Geometric rules are generated at all levels which compete with the hierarchical rule.

3. The hierarchical model

Define on each site of the Sierpinski gasket a site occupation variable n_i. The sites are occupied randomly with a concentration $\langle n_i \rangle = p$. To define a percolation model we must have rules for determining when a given set of occupied sites constitutes a percolation cluster. The problem then is to determine the critical concentration p_c, at which there first exists a percolation cluster of infinite size.

The first model we call the $m = 3$ model. Here the three sites of an elementary up-triangle (referred to as a T_0-unit; see figure 1) are considered as contained in the same cluster when all three of these sites are occupied, otherwise not. The six sites of the T_1-unit (see figure 1) are in the same cluster if all are occupied. This is equivalent to saying that all three of the T_0-subunits that compose T_1 must be percolated in order that T_1 be considered percolated. This procedure is repeated so, for example, a T_{n+1}-unit is percolated if all three of its T_n-subunits are percolated, and so on. Clearly, there exists a percolating cluster spanning the lattice only for $p_c(3) = 1$ since by the $m = 3$ rules all of the sites on the lattice must be present for such a cluster to exist.

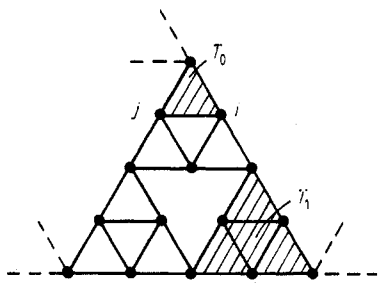

Figure 1. A T_2-unit section taken from an infinite Sierpinski gasket. The sites of the gasket are labelled $i, j, \ldots$. The T_0-units are the elementary up-pointing triangles, the T_1-units are composed of three T_0-units making a larger up-triangle, and so on.

The $m = 2$ model is defined in a similar fashion. Here the condition for a T_{n+1}-unit to be percolated is that at least two of its three T_n-subunits must be percolated. In this case a site is contained in the T_{n+1} percolation cluster only if it directly contributes to the percolation of the T_n-subunits, and so on. As opposed to the $m = 3$ model, this model makes it much easier for a given configuration of occupied sites to form a percolation cluster. Therefore we should expect that $p_c(2) < 1$.

In each case, as we vary p, we vary the average size and average distribution in size of the percolation clusters on the lattice. These rules therefore define two hierarchical percolation models: $m = 3$, where three percolated T_{n-1}-subunits imply a percolated T_n-unit and $m = 2$, where at least two percolated T_{n-1}-subunits imply a percolated T_n-unit.

We now construct a renormalisation transformation R such that

$$R\colon T_1(p) = T_0(p') \tag{1}$$

for every T_1-unit on the infinite lattice. (Here $T_1(p)$ implies that each site of the T_1-unit is occupied with a probability p.) This is equivalent to tracing over the three internal sites of each T_1-unit. From this we obtain a recursion relation for the site occupation probability

$$p' = p'(p) \tag{2a}$$

with a scale factor $b=2$. The correlation length exponent is given by

$$\frac{1}{\nu}=\frac{\ln(\mathrm{d}p'/\mathrm{d}p)_{p_c}}{\ln(2)}. \tag{2b}$$

Due to the fact that the T_0-units share underlying sites this direct transformation is difficult to perform. Matters are made simple by considering a Sierpinski gasket of finite size $M_n(p)$. Here $M_n(p)$ is equivalent to a T_n-unit taken by itself in which each site is occupied with probability p. We then operate upon $M_n(p)$ n times with the renormalisation transformation R thus reducing it to a single site with a total renormalised site probability p_t, which is equivalent to the probability that there is a spanning cluster on the finite size lattice. We then have the equation

$$R^n\colon M_n(p)=R^{n-1}\colon M_{n-1}(p')=p_t \tag{3}$$

which in the limit n taken to infinity is equivalent to the renormalisation transformation defined by (1). This defines the recursion relation (2). Equation (3) therefore represents a small-cell renormalisation transformation. The idea is to solve this equation as an explicit function of n and then take the limit n to infinity [13].

For the $m=3$ model the result is rather simple. Since all of the sites of the gasket must be occupied in order that there be a spanning percolation cluster, there is only one configuration to consider, and so

$$R^n\colon M_n(p)=p^{r_n} \tag{4}$$

where r_n, the number of sites on a Sierpinski gasket M_n, is

$$r_n=2+3^n\left(1+\frac{1}{3}+\ldots+\frac{1}{3^n}\right)\to 3^n \qquad n\to\infty. \tag{5}$$

Using (3), we find that our recursion relation has the explicit form

$$p'=p^{(r_n/r_{n-1})} \tag{6}$$

which shows that $p_c=1$ for the critical concentration. For the correlation length exponent we have

$$\frac{1}{\nu}=\lim_{n\to\infty}\frac{\ln(r_n/r_{n-1})}{\ln(2)}=\frac{\ln(3)}{\ln(2)}=1.585\ldots=D \tag{7}$$

where D is the fractal dimension of the Sierpinski gasket. Thus the correlation length diverges with a power law: $\xi\propto|1-p|^{-1/D}$. This should be compared with the corresponding result for geometric percolation on the Sierpinski gasket where one finds [14]: $p_c=1$ and $\xi\propto \mathrm{e}^{(1-p)^{-2}}$. Although neither model exhibits a percolation transition on the Sierpinski gasket, in the limit p to unity they exhibit distinct forms for the divergence of the correlation length. This therefore places the two models in distinct universality classes.

For the $m=2$ model, write the total renormalised site probability as

$$p_t=\sum_{k=0}^{r_n}\alpha_n(k)p^{r_n-k}q^k \tag{8}$$

where $q=1-p$. p_t is the total probability that the lattice $M_n(p)$ possesses a percolating configuration. $\alpha_n(k)$ is the number of percolating configurations in which k sites are unoccupied. Furthermore, since

$$1=(p+q)^{r_n}=p_t+(1-p_t) \tag{9}$$

we have

$$1-p_t=\sum_{k=0}^{r_n}\alpha'_n(k)p^{r_n-k}q^k \tag{10}$$

with

$$\alpha_n(k)+\alpha'_n(k)=\begin{Bmatrix} r_n \\ k \end{Bmatrix} \tag{11}$$

where $\alpha'_n(k)$ is the number of non-percolating configurations with k sites unoccupied and $\begin{Bmatrix} r_n \\ k \end{Bmatrix}$ is the binomial coefficient. Then, from the property of the binomial coefficients,

$$\sum_{k=0}^{r_n}(\alpha_n(k)+\alpha'_n(k))=\sum_{k=0}^{r_n}\begin{Bmatrix} r_n \\ k \end{Bmatrix}=2^{r_n}. \tag{12}$$

Under the substitution $p\to q$, percolating configurations with k unoccupied sites map onto non-percolating configurations with (r_n-k) unoccupied sites (and vice versa). Since this mapping is unique,

$$\alpha_n(k)=\alpha'_n(r_n-k) \tag{13}$$

and therefore

$$\sum_{k=0}^{r_n}\alpha_n(k)=\sum_{k=0}^{r_n}\alpha'_n(k)=2^{r_n-1}. \tag{14}$$

The critical concentration is obtained from the fixed point of (2). Using (8) and the fact that we expect only one percolation transition in this model, the unstable fixed point is

$$p_t(p=\tfrac{1}{2})=(\tfrac{1}{2})^{r_n}\sum_{k=0}^{r_n}\alpha_n(k)=\tfrac{1}{2} \tag{15}$$

which identifies $p_c=\frac{1}{2}$ as the critical concentration exactly, independent of the value of n. The remaining two trivial (stable) fixed points are located at $p=0$ and $p=1$.

For the correlation length exponent, one can work out recursion relations to calculate the coefficients $\alpha_n(k)$ for a specific value of n given the values at $n-1$. Using (3) in the form

$$\sum_{k=0}^{r_{n+1}}\alpha_{n+1}(k)p^{r_{n+1}-k}q^k=\sum_{k=0}^{r_n}\alpha_n(k)(p')^{r_n-k}(q')^k \tag{16}$$

one finds that at $p_c=\frac{1}{2}$

$$\frac{1}{\nu_n}=\frac{\ln(\gamma_n)}{\ln(2)} \tag{17a}$$

where

$$\gamma_n=\left[2^{1-r_{n+1}}\left(r_{n+1}\sum_{k=0}^{r_{n+1}}\alpha_{n+1}(k)-2\sum_{k=0}^{r_{n+1}}k\alpha_{n+1}(k)\right)\right]\times\left[2^{1-r_n}\left(r_n\sum_{k=0}^{r_n}\alpha_n(k)-2\sum_{k=0}^{r_n}k\alpha_n(k)\right)\right]^{-1} \tag{17b}$$

and upon numerical evaluation of these coefficients one finds that to $n=2$, $1/\nu_2 = 0.565\ldots$. I conjecture that the exact value is given by

$$\frac{1}{\nu} = D - 1 = 0.585\ldots \tag{18}$$

which, as we will see presently, is supported by considerations of a modified hierarchical model. The $m=2$ model therefore possesses a non-trivial percolation transition with a correlation length exponent that differs from both the $m=3$ and the geometric models. This is a clear indication that the $m=2$ model is in a distinct universality class from either of these models on the Sierpinski gasket.

4. The modified hierarchical model

Define on each site of the Sierpinski gasket two occupation variables (n_i, m_i) where the i-index labels the site and each variable is associated with one of the two elementary T_0-units connected to site i. With $n_i = 0, 1$ and $m_i = 0, 1$ each site can be in four states of relative occupation. Assume random occupation with $\langle n_i \rangle = \langle m_i \rangle = p$. The rules for defining percolation clusters are exactly as before. A T_0-unit is considered percolated if, for $m=3$, all three of its occupation variables are occupied and for $m=2$, if at least two of the three are occupied, and so on. The important feature of this modified model is that the elementary T_0-units are decoupled (they each have their own set of occupation variables), while at the same time the hierarchical nature of the percolation clusters is preserved. Therefore, we attempt to determine whether the coupling between the T_0-units in the previous model is relevant in determining its critical properties.

We may now directly perform the renormalisation transformation of (1) as illustrated in figure 2. Here we associate the 'total' renormalised site probability of a given T_0-unit with the site that is not traced over in the single application of the renormalisation transformation. For the two models under consideration we have for the recursion relations, with scale factor $b=2$,

$$m=3 \qquad p' = p^3 \tag{19a}$$

$$m=2 \qquad p' = 3p^2 - 2p^3 \tag{19b}$$

so for $m=3$ we find $p_c = 1$, $1/\nu = D$ and for $m=3$ we find $p_c = \frac{1}{2}$, $1/\nu = D-1$. These results are exactly what we found for the hierarchical model of § 2 and support the

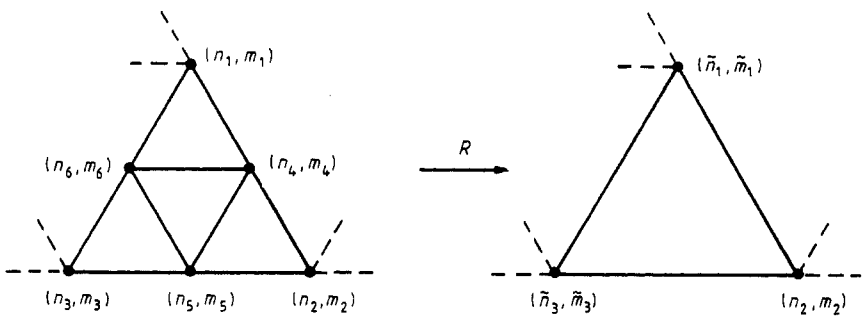

Figure 2. The renormalisation transformation for the modified hierarchical model. We write $n_i = \langle n_i \rangle$, $m_i = \langle m_i \rangle$ for simplicity.

conjecture that $1/\nu = D-1$ is exact for that $m=2$ model. This model is therefore in the same universality class as the former model. The important result here is that the coupling of the T_0-units is irrelevant (in the renormalisation group sense). Therefore it is the hierarchical nature of the percolation clusters that constitutes the relevant feature of the model and which distinguishes it from the geometric percolation model.

5. Formulation on the triangular lattice

When the hierarchical model is formulated on the triangular lattice it becomes possible for two percolated clusters to share more than a single site (a single shared site is the maximum allowed on the Sierpinski gasket). For example, percolated T_n and T_m units located on distinct Sierpinski gaskets embedded within the triangular lattice may overlap in a T_r-unit for some $r \geq 0$ (figure 3(c)). This additional connectivity between the hierarchically structured percolation clusters is the new property introduced on the triangular lattice. The question is whether this new property leads to a change in the relevant features of the hierarchical model.

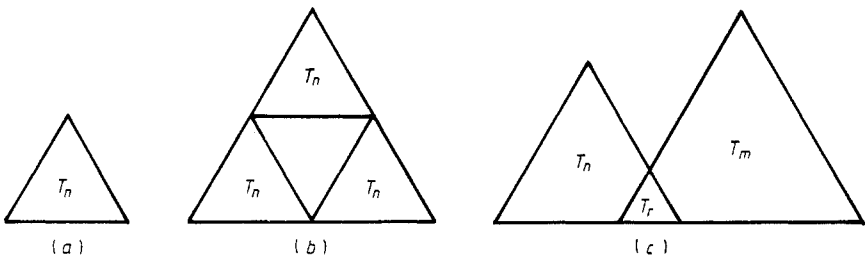

Figure 3. The basic connection rules for the hierarchical model on the triangular lattice. In (c) $0 \leq r \leq \min(n, m)$.

To understand how we treat this new property, we work in analogy with the geometric percolation interpretation for the $T=0$ properties of the site-diluted Ising ferromagnet.

Imagine that defined on each occupied site is a spin variable S_i. From the $\{S_i\}$ construct a lattice spin Hamiltonian $H(\{S_i\})$ whose ordered states (which are assumed degenerate in energy) correspond to the hierarchical percolation clusters when restricted to the Sierpinski gasket. The percolation model is then a model for the $T=0$ structure of this site-diluted lattice spin model. The percolated clusters are interpreted as regions in which there is a well defined local order imposed by the Hamiltonian via the $T=0$ energy minimisation constraint. A group of clusters are considered to coalesce if they share 'sufficient' constraints: either directly by the hierarchical rules defined in § 2, or indirectly by shared regions of well defined order. The indirect constraint applies when hierarchically defined clusters on distinct embedded Sierpinski gaskets share a region with a well defined local order and by consistency (i.e. energy minimisation) this same order is imposed on those clusters within which this region is contained (i.e. domain walls between different possible orderings on the same percolation cluster cost energy). We assume throughout that a shared site is an insufficient constraint for clusters to coalesce (as in the formulation on the Sierpinski gasket), however a shared T_r-unit for $r \geq 0$ is sufficient.

Based on the above considerations, we formulate specific realisations of the $m=3$ and $m=2$ models that reduce to their usual form on the Sierpinski gasket. In figure 3 are shown the three basic rules for percolation cluster construction. Rules $3(a)$ and $3(b)$ are the usual ones defined for the Sierpinski gasket formulation. In $3(a)$ the T_n-unit is considered percolated according to the $m=2$ or $m=3$ rules and in $3(b)$ we see the construction of a T_{n+1}-unit, again according to the $m=2$ or $m=3$ rules. These two rules, therefore, are responsible for imposing the hierarchical structure on the percolation clusters.

The new element here is figure $3(c)$. For $m=3$ this states that if T_n and T_m are percolated (i.e. fully occupied), then as long as they share at least a mutual T_0-unit, their sum is considered a single percolation cluster. A few examples of the new types of percolation clusters so introduced are illustrated in figure 4 for the $m=3$ model. For the $m=2$ model this rule states that as long as the mutual T_r-unit is both percolated and contributes to the percolation of both the T_n and T_m units, then their sum constitutes a single percolation cluster. We emphasise that rule $3(c)$ is not present on the Sierpinski gasket since the ordered regions can share at most one site and therefore order is propagated solely by the hierarchical rules $3(a)$ and $3(b)$. The present formulation therefore reduces to the previous one when defined on the Sierpinski gasket.

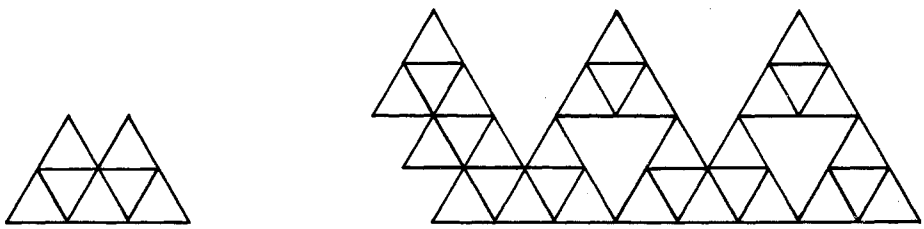

Figure 4. Examples of percolation clusters for the $m=3$ model defined on the triangular lattice. The smallest up-triangle drawn may be considered T_n-units for $0 \leq n$.

Here we present some preliminary considerations for the $m=3$ model. First one can show that $p_c(3)<1$ and so there is an upper bound to the critical concentration. Consider the rule defined in figure $4(a)$ restricted in application to the T_1-units (furthermore ignore the hierarchical rules beyond the construction of the T_1-units from the T_0-units, and the T_0-units from the underlying sites). By restricting ourselves only to this rule, we ignore all of the longer range connections which, however, would only make it easier to percolate. Therefore the p_c of this restricted model places an upper bound on the p_c of the full model.

Since we occupy the sites of the triangular lattice at random with probability p, a given T_1-unit is occupied with probability $P_{T_1}(p)=p^6$ (i.e. it contains six sites). Furthermore, considering the down-triangle located at the centre of a T_1-unit as occupied if its associated T_1-unit is fully occupied, we see that this is essentially a nearest-neighbour geometric percolation model for these down-triangles (note that the down triangles themselves form a triangular lattice). That this model is only approximately a geometric percolation model comes from the fact that the T_1-units are not distributed completely randomly (a detailed discussion of this can be found in [12]). However, the correlations in the T_1-distribution can only lead to a suppression of the

critical concentration. For the nearest-neighbour geometric percolation model the critical concentration is known to be exactly [15] $p_c = \frac{1}{2}$, which implies that $P_{T_1}(p_c) \leqslant \frac{1}{2}$ or $p_c \leqslant 0.891$. Monte Carlo calculations on this restricted model indicate that [12] $p_c = 0.88 \pm 0.02$ and the calculated exponents are numerically consistent with those of the geometric percolation model.

We have shown that p_c for the full $m = 3$ model is bounded from above and for a restricted model we find geometric percolation exponents. These results indicate that the $m = 3$ model may be contained in the geometric percolation universality class, though we have only argued from the lowest order (i.e. shortest range) in the connection rules. Furthermore we have not ruled out the possibility that $p_c = 0$ due to the presence of long range (and possibly infinite range) connections. For example, in the geometric percolation model with nth-neighbour connections for all n (including $n = \infty$) the percolation order parameter has the exact form $P(p) = p$, thus $p_c = 0$, since all occupied sites are always contained in one cluster (also $\beta = 1$, which is the mean-field/Bethe lattice result). An argument against this possibility for the $m = 3$ model is the following: the only manner in which an infinite range connection is possible in this model is between two infinite Sierpinski gaskets. Since the existence of an infinite size Sierpinski gasket implies the existence of three rows of sites that are completely occupied, for $p \neq 1$ this has negligible probability. So there are no infinite range connections and therefore the claim is that $p_c > 0$.

In summary, there are indications that the $m = 3$ model formulated here is contained in the geometric percolation universality class, though the approximations made leave out much of the structure of the full model. The $m = 2$ model is much more complex, though the arguments for the upper bound to p_c in the $m = 3$ model also hold here since it can only be easier to percolate in the $m = 2$ model.

6. Conclusion

We have presented here two versions of a hierarchical percolation model on the Sierpinski gasket and some exact results for the percolation properties have been obtained. These results show that on the Sierpinski gasket the hierarchical percolation model is in a distinct universality class from that of the geometric percolation model on the same lattice. Furthermore, through consideration of a modified hierarchical model we have identified an irrelevant aspect of the original model, showing that it is the hierarchical nature of the percolation clusters that is relevant. Finally the hierarchical model was formulated on the two-dimensional triangular lattice and some preliminary results indicate that the increased connectivity introduced on this lattice is relevant for the $m = 3$ model and, furthermore, that it may be contained in the geometric percolation universality class.

Acknowledgments

I thank M Schick for many useful suggestions and discussions and for a careful reading of preliminary drafts of this paper. I also thank G Forgas for helpful discussions. This work was supported in part by the National Science Foundation under Grant DMR-8613598.

References

[1] Yeomans J M and Stinchcome R B 1979 *J. Phys. C: Solid State Phys.* **12** 347
[2] Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
[3] Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 536
[4] Wu F 1978 *J. Stat. Phys.* **18** 115
[5] Blease J 1977 *J. Phys. C: Solid State Phys.* **10** 917
[6] Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
[7] Chalupa J, Leath P L and Reich G R 1979 *J. Phys. C: Solid State Phys.* **12** L31
[8] Adler J and Aharony A 1988 *J. Phys. A: Math. Gen.* **21** 1425
[9] Adler J, Palmer R G and Meyer H 1987 *Phys. Rev. Lett.* **58** 882
[10] Adler J, Gefen Y, Schick M and Shih W H 1987 *J. Phys. A: Math. Gen.* **20** L227
[11] Fried H and Schick M 1988 *Phys. Rev.* B **38** 954
[12] Fried H and Schick M 1989 *Phys. Rev.* B to appear
[13] Reynolds P J, Stanley H E and Klein W 1980 *Phys. Rev.* B **21** 1223
[14] Gefen Y, Aharony A, Shapir Y and Mandelbrot B 1984 *J. Phys. A: Math. Gen.* **17** 435
[15] Sykes M F and Essam J W 1964 *Phys. Rev.* A **133** 310